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2004 J. Phys. A: Math. Gen. 37 3599

(http://iopscience.iop.org/0305-4470/37/11/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 3599–3606 PII: S0305-4470(04)65741-6

Relationship between dwell, transmission and reflection tunnelling times

Mario Goto[1], **Hiromi Iwamoto**[1], **Veríssimo M de Aquino**[1], **Valdir C Aguilera-Navarro**[2] **and Donald H Kobe**[3]

[1] Departamento de Física/CCE, Universidade Estadual de Londrina, 86051-990 Londrina, PR, Brazil
[2] Departamento de Química e Física, Universidade Estadual do Centro-Oeste, 85015-430 Guarapuava, PR, Brazil
[3] Department of Physics, University of North Texas, Denton, TX 76203, USA

Received 8 July 2003, in final form 29 January 2004
Published 2 March 2004
Online at stacks.iop.org/JPhysA/37/3599 (DOI: 10.1088/0305-4470/37/11/005)

Abstract

The transmission tunnelling time through a barrier is often defined as the probability that the particle is in the barrier divided by the *transmission* probability current density. The dwell time, the time the particle stays in the barrier, is commonly defined as the probability that the particle is in the barrier divided by the *incident* probability current density. We define the reflection tunnelling time in a similar way by dividing the probability that the particle is in the barrier by the *reflection* current density. By conservation of probability, the reciprocal of the dwell time is equal to the sum of the reciprocals of the transmission and reflection times. This relation is illustrated by calculating the different tunnelling times for a rectangular barrier.

PACS numbers: 03.65.−w, 03.65.Xp, 73.40.Gk

1. Introduction

Since the early days of quantum mechanics the question has been asked: 'How long does a particle spend in a barrier during tunnelling?' Two other questions related to this one may also be asked: 'How long does a particle spend in the barrier before being transmitted?' and 'How long does a particle spend in the barrier before being reflected?' The duration of a particle in the barrier regardless of how it escapes is called the *dwell* time. The time it takes for a particle to enter the barrier and be transmitted is called the *transmission* tunnelling time. The time it takes for a particle to enter the barrier and be reflected is called the *reflection* tunnelling time [1–4]. The relationship between these three times is still another question, an answer to which is proposed in this paper.

The *transmission* tunnelling time for a stationary state can be defined as the reciprocal *velocity field* of the particle integrated over the length of the barrier. The velocity field of the particle is the probability current density of the particle in the barrier divided by the probability of finding the particle in the barrier [5–7]. From conservation of probability the probability current density in the barrier is a constant equal to the transmitted probability current density. Therefore, the transmission tunnelling time is equal to the probability of finding the particle in the barrier divided by its transmission current density. The *dwell* time is commonly defined as the probability of finding the particle in the barrier divided by its incident probability current density [10, 13]. We define the *reflection* tunnelling time in a similar manner as the probability of finding the particle in the barrier divided by its reflection current density. The relationship between these three times, based on conservation of probability, is that the reciprocal of the dwell time is equal to the sum of the reciprocals of the transmission and reflection times. As an example, we calculate the transmission time, reflection time and dwell time for a rectangular barrier and demonstrate their relationship.

In section 2 the transmission tunnelling time is defined. The dwell time is defined in section 3. In section 4 an analogous definition is given for the reflection tunnelling time. In section 5 we show from conservation of probability that the reciprocal of the dwell time is equal to the sum of the reciprocals of the transmission and reflection times. As an example these tunnelling times are calculated in section 6 for a rectangular potential. The conclusion is given in section 7.

2. Transmission tunnelling time

The transmission tunnelling time τ_t, or just transmission time, of a quantum particle in one dimension is the time it takes for a particle to traverse the barrier. It is defined as the reciprocal velocity field of the particle integrated over the width of the barrier. The velocity field of the particle is its probability current density divided by its probability density.

The probability density $\rho = \rho(x, t)$ at the point x and at time t of a quantum particle with a (normalized) wavefunction $\psi = \psi(x, t)$ is its absolute value squared, $\rho = |\psi|^2$. The probability current density j of a particle is the real part of its velocity operator p/m sandwiched between its (normalized) wavefunctions:

$$j(x, t) = \mathrm{Re}\, \psi^* \frac{p}{m} \psi \tag{2.1}$$

where the canonical momentum operator is $p = -\mathrm{i}\hbar\partial/\partial x$, the mass of the particle is m and the real part is denoted by Re. Probability is conserved locally, so the probability density ρ and the probability current density j together satisfy the equation of continuity, $\partial\rho/\partial t + \partial j/\partial x = 0$.

The velocity field $v(x, t)$ for the 'probability fluid' of the particle in terms of its probability density and probability current density is

$$v(x, t) \equiv \frac{j(x, t)}{\rho(x, t)} \tag{2.2}$$

in analogy with the classical velocity field for a fluid. In this ratio the normalization constants for the wavefunctions in the numerator and denominator cancel, so unnormalized wavefunctions can be used for the velocity field. We will set this velocity field equal to $\mathrm{d}x(t)/\mathrm{d}t$ as in the de Broglie–Bohm formulation of quantum mechanics [12].

In the de Broglie–Bohm approach to tunnelling times, the velocity field in equation (2.2) for a time-dependent wave packet is proportional to the partial derivative of the phase of the wavefunction with respect to x. This velocity field is then set equal to $\mathrm{d}x(t)/\mathrm{d}t$ and the equation is solved for the quantum trajectory $x = x(t)$ assuming an initial displacement $x(0)$. The time it takes for the particle to tunnel through the barrier is then calculated. These times are averaged over all the initial displacements corresponding to the initial wave packet [7, 14] to obtain an average tunnelling time.

Our approach is to consider a stationary state of a particle with any shape of potential barrier $V(x)$ with finite support, i.e., $V(x) = 0$ for $\{x|x < 0, x > a\}$ for some a. For a stationary state (normalized) wavefunction $\psi = \psi(x)$ corresponding to an energy E less than the height of the barrier, the probability density $\rho(x) = |\psi(x)|^2$ is time independent. From the equation of continuity the probability current density j is constant. Therefore, by conservation of probability, the probability current density $j = j_B$ in the barrier $B = \{x|V(x) > E\}$ is equal to the transmitted probability current density j_t in the region $x > a$ beyond the barrier. The time-independent velocity field $v(x) = j_B/\rho(x) = j_t/\rho(x)$ in the barrier is equal to the velocity $\mathrm{d}x/\mathrm{d}t$ in the barrier, so $\mathrm{d}t = v(x)^{-1}\,\mathrm{d}x$. The *transmission* tunnelling time, or just transmission time, τ_t for the particle is thus given by the common expression [5, 12]

$$\tau_t = \int_B \frac{\mathrm{d}x}{v(x)} = \frac{1}{j_t}\int_B \mathrm{d}x\,\rho(x) \tag{2.3}$$

where the integration is over the barrier region B and the integral in equation (2.3) is the probability of finding the particle in the barrier [2, 24].

For any shape of the potential barrier with finite support a particle of energy E and mass m in the region to the left of the barrier ($x < 0$) has an incident wavefunction $\psi_i(x) = \exp(\mathrm{i}kx)$, where the wave number is $k = (2mE)^{1/2}/\hbar$. Without loss of generality the incident amplitude can be taken as unity. (Normalization in a box of unit volume can be used.) The particle also has a wavefunction to the right of the barrier ($x > a$) that is $\psi(x) = D\exp(\mathrm{i}kx)$, where D is the complex transmission amplitude. From equation (2.1) the probability current density in the transmitted region is

$$j_t = \frac{\hbar k}{m}T \tag{2.4}$$

where $T = |D|^2$ is the *transmission* coefficient. If equation (2.4) is substituted into equation (2.3) the transmission time is [5]

$$\tau_t = \frac{m}{\hbar k T}\int_B \mathrm{d}x\,\rho(x). \tag{2.5}$$

For a fixed probability of the particle being in the barrier region, there is an inverse relationship between the transmission coefficient T and the transmission time τ_t. The less probable it is for the particle to be transmitted, the longer it takes for the particle to traverse the barrier. If the transmission coefficient were zero, it would take an infinite time for the particle to traverse the barrier. In other words, if it were impossible for a particle to be transmitted, the corresponding time for transmission to occur would be infinite. Conversely, if the transmission coefficient were unity, there would be no reflection. The transmission time and the dwell time would be the same in this case.

3. Dwell time

The *dwell time* is the time that the particle spends in the barrier region, regardless of how it escapes [1]. The particle can escape from the barrier by one of two channels—either transmission or reflection. The dwell time [10, 11] is taken here as the probability for the particle to be in the barrier divided by the *incident* probability current density j_i,

$$\tau_d = \frac{1}{j_i}\int_B \rho(x)\,\mathrm{d}x. \tag{3.1}$$

This expression was derived for a wave packet by Leaven and Aers [13] and is given in appendix B of [3].

If the incident wavefunction is a plane wave $\psi_i(x) = \exp(\mathrm{i}kx)$, the incoming current density is

$$j_i = \frac{\hbar k}{m} \tag{3.2}$$

from equation (2.1). The dwell time is thus given by the common expression

$$\tau_d = \frac{m}{\hbar k} \int_B \rho(x)\,\mathrm{d}x. \tag{3.3}$$

If the barrier width goes to zero for a finite potential, the probability of the particle to be in the barrier goes to zero and the dwell time is zero.

4. Reflection time

The *reflection tunnelling time*, or just reflection time, τ_r is the time it takes for a particle to be reflected out of the barrier. We define it in a way analogous to the definition of the transmission tunnelling time and the dwell time. The reflection tunnelling time τ_r is defined as the probability for the particle to be in the barrier divided by the magnitude of the reflection probability current density,

$$\tau_r = \frac{1}{|j_r|} \int_B \mathrm{d}x\, \rho(x). \tag{4.1}$$

This definition follows from the conservation of probability and the previous definitions of the transmission and dwell times.

For the same potential barrier, the same particle of energy E in the region to the left of the barrier ($x < 0$) has a wavefunction of the form $\psi(x) = \exp(\mathrm{i}kx) + A\exp(-\mathrm{i}kx)$, where A is the complex reflection amplitude. From equation (2.1) the reflection current density is

$$j_r = -\frac{\hbar k}{m} R \tag{4.2}$$

where $R = |A|^2$ is the reflection coefficient. The reflected current density is directed to the left. If equation (4.2) is substituted into equation (4.1), the reflection tunnelling time is

$$\tau_r = \frac{m}{\hbar k R} \int_B \mathrm{d}x\, \rho(x). \tag{4.3}$$

For a fixed probability of the particle being in the barrier region, there is an inverse relationship between the reflection coefficient R and the reflection time τ_r. The less probable it is for the particle to be reflected, the longer it takes for the particle to be reflected. If the reflection coefficient were zero, as it would be in resonant tunnelling, it would take an infinite time for the particle to be reflected. In other words, if it were impossible for a particle to be reflected, the corresponding time for reflection to occur would be infinite. Conversely, if the reflection coefficient were unity, there would be no transmission. The reflection time and the dwell time would be the same in this case.

5. Relationship between tunnelling times

5.1. Dwell, transmission and reflection times

The incident probability current density j_i is equal to the sum of the outgoing current densities

$$j_i = j_t + |j_r| \tag{5.1}$$

because probability is conserved and there are no sources or sinks of probability. From equations (2.4), (4.2) and (3.2), equation (5.1) gives the condition that the sum of the transmission coefficient T and the reflection coefficient R is unity, $R + T = 1$.

If we divide equation (5.1) by the probability of finding the particle in the barrier $\int_B \mathrm{d}x\, \rho(x)$, we find from equations (3.1), (2.3) and (4.1) a relationship between the dwell, transmission and reflection tunnelling times,

$$\frac{1}{\tau_d} = \frac{1}{\tau_t} + \frac{1}{\tau_r}. \tag{5.2}$$

Thus the dwell time is less than the transmission time and the reflection time separately, since two channels are available for the particle to escape from the barrier. This expression is a new relationship between the three times. For resonant tunnelling the reflection time would be infinite and equation (5.2) would imply that the dwell time would equal the transmission time.

Based on the definitions of the dwell, transmission and reflection times in equations (3.3), (2.5) and (4.3), we can see that these times satisfy the relationship

$$\tau_d = T\tau_t = R\tau_r. \tag{5.3}$$

Therefore, we have the inequalities $\tau_d \leqslant \tau_t$ and $\tau_d \leqslant \tau_r$ because the particle in the barrier can escape by either transmission or reflection. If the reflection time is equal to the transmission time $\tau_r = \tau_t$, then $R = T = 1/2$ and equation (5.3) gives $2\tau_d = \tau_r = \tau_t$. The dwell time is half the transmission time or the reflection time in this case, since the particle can escape the barrier region by either transmission or reflection. Equation (5.3) is a consequence of our definitions and is in contrast to another relation that has been suggested.

5.2. Another relation

Another relationship between the dwell time τ_d', the reflection time τ_r' and the transmission time τ_t' that has been suggested is

$$\tau_d' = R\tau_r' + T\tau_t' \tag{5.4}$$

where the primes denote that the times are in general defined differently from those above [3, 7, 16–22]. This relation should be compared with equations (5.2) and (5.3). Equation (5.4) has been criticized by a number of authors [2, 24, 26]. In particular it neglects the possibility of interference terms.

If $\tau_r' = \tau_t'$, then both the reflection time and the transmission time are equal to the dwell time τ_d'. The dwell time τ_d' is the time the particle remains in the barrier, regardless of whether it is transmitted or reflected. Since there are two channels for escaping the barrier, it seems strange that in this case the dwell time would be the same as the other times. Equation (5.4) is obviously inconsistent with equation (5.3) for the dwell, transmission and reflection times defined above (without primes) that satisfy equation (5.2).

Equation (5.4) involves quantities other than the tunnelling times, namely, R and T, that must be calculated or measured. With experimental data on the transmission, reflection and dwell times, as well as reflection and transmission coefficients, it will be possible to see which, if either, of the two relations in equations (5.2) and (5.4) is valid.

6. Rectangular potential barrier

As an example, the tunnelling times are now calculated for a rectangular barrier and their relationship is shown. The potential energy for a rectangular barrier in one dimension is

$V(x) = V_0$ for $0 < x < a$ and 0 otherwise. The solution to the time-independent Schrödinger equation for an energy less than the barrier height $E < V_0$ is the wavefunction

$$\psi(x) = \begin{cases} \exp(\mathrm{i}kx) + A\exp(-\mathrm{i}kx) & \text{for} \quad x < 0 \\ B\exp(-\kappa x) + C\exp(\kappa x) & \text{for} \quad 0 \leqslant x \leqslant a \\ D\exp(\mathrm{i}kx) & \text{for} \quad x > a \end{cases} \tag{6.1}$$

where the incident, reflected and transmitted wave number is $k = (2mE)^{1/2}\hbar^{-1}$ and the imaginary wave number in the barrier is $\mathrm{i}\kappa = \mathrm{i}[2m(V_0 - E)]^{1/2}\hbar^{-1}$. A characteristic wave number in this problem is $k_0 = (2mV_0)^{1/2}\hbar^{-1}$, so $k^2 + \kappa^2 = k_0^2$. The incident wavefunction $\psi_i(x) = \exp(\mathrm{i}kx)$ in equation (6.1). The amplitudes in the wavefunction in equation (6.1) are determined from the boundary conditions that the wavefunction and its derivative are continuous at 0 and a. The rectangular barrier is discussed in most quantum mechanics textbooks, see e.g. [23].

From equation (2.5) the transmission tunnelling time is

$$\tau_t = (4k\kappa^3)^{-1}Q \tag{6.2}$$

where

$$Q = \sinh(2\kappa a) - 2\kappa a(k^2 - \kappa^2). \tag{6.3}$$

The tunnelling times are measured in units of a characteristic time $\tau_0 = \hbar/2V_0$, length is measured in units of k_0^{-1} and energy is measured in units of V_0.

From equation (4.3) the reflection time τ_r is

$$\tau_r = k\kappa^{-1}\sinh^{-2}(\kappa a)Q \tag{6.4}$$

and from equation (3.3) the dwell time τ_d is

$$\tau_d = k\kappa^{-1}[\sinh^2(\kappa a) + (2\kappa k)^2]^{-1}Q. \tag{6.5}$$

Together, the sum of the reciprocals of τ_t and τ_r in equations (6.2) and (6.4), respectively, gives the reciprocal of τ_d in equation (6.5). Therefore, equation (5.2) for the relationship between the tunnelling times is explicitly verified.

A simple numerical application can be made using the potential parameters for the heterostructure GaAs/$Al_{0.3}Ga_{0.7}$As/GaAs [28]. The potential height is $V_0 \simeq 0.3$ eV, a typical width is $a \simeq 50$ Å and the effective mass is $m_{\mathrm{eff}} \simeq 0.067$ m. For these values of the relevant physical quantities the dimensionless width of the potential is $a \simeq 3.63$ in units of $k_0^{-1} = 13.77$ Å and the characteristic time is $\tau_0 \simeq 1.1$ fs. The tunnelling times τ_t, τ_r and τ_d are shown in figure 1 as a function of energy E at a fixed barrier width a, and in figure 2 as a function of barrier width a at fixed energy E.

In figure 1 the transmission time τ_t in equation (2.5) decreases with increase in energy, since the transmission coefficient T in the denominator increases with energy [23]. At zero energy there is no transmission $T = 0$, so the transmission time is infinite as expected for a process that does not occur. Conversely, the reflection time τ_r in equation (4.3) increases with increase in energy, since the reflection coefficient R in the denominator decreases with increase in energy. On the other hand, the dwell time τ_d in equation (6.5) approaches zero as the energy approaches zero because there is very little penetration of the particle into the barrier region. The dwell time slowly varies as the energy approaches unity because the transmission and reflection times for the rectangular barrier in equation (5.2) compensate each other.

In figure 2 the transmission time τ_t in equation (2.5) increases with increase in barrier thickness a, since the transmission coefficient T in the denominator decreases with the barrier thickness. For zero thickness the transmission coefficient is unity, so the transmission time approaches the dwell time as the thickness decreases. Conversely, the reflection time τ_r in

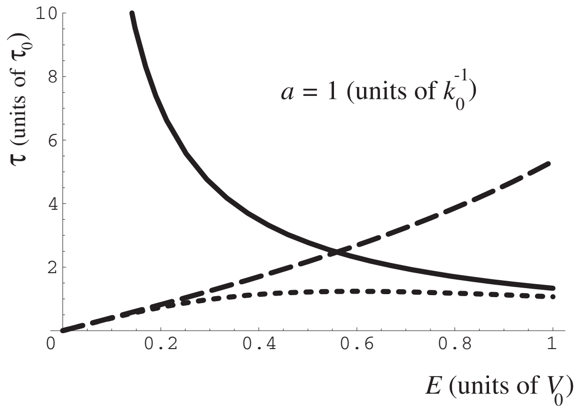

Figure 1. Tunnelling times τ (in units of τ_0) versus energy E (in units of V_0) for the rectangular barrier with fixed width $a = 1$ (in units of k_0^{-1}). The transmission time τ_t is the solid line, the reflection time τ_r is the dashed line and the dwell time τ_d is the dotted line.

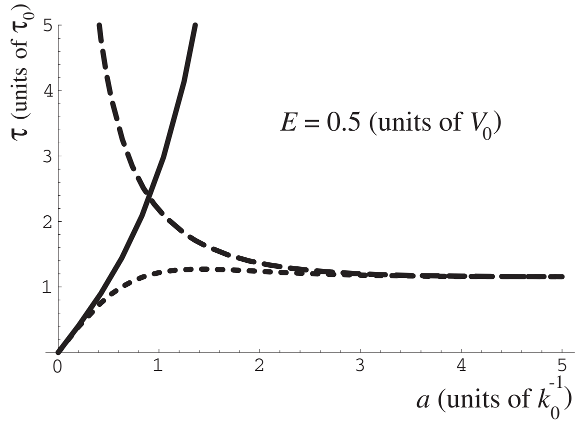

Figure 2. Tunnelling times τ (in units of τ_0) versus rectangular barrier width a (in units of k_0^{-1}) for fixed energy $E = 0.5$ (in units of V_0). The transmission time τ_t is the solid line, the reflection time τ_r is the dashed line and the dwell time τ_d is the dotted line.

equation (4.3) decreases with increasing thickness, since the reflection coefficient R in the denominator increases with increase in thickness. On the other hand, the dwell time τ_d approaches zero as the thickness approaches zero because in this limit there is no barrier. The dwell time τ_d approaches a constant for large barrier thickness, which is an example of the Hartman effect [29]. In [33] this Hartman effect also appears in a phase time [30–32].

7. Conclusion

Further experimental measurements of dwell, transmission and reflection times, along with reflection and transmission coefficients, using electrons and photons [34–36] are highly desirable. All the proposals for these tunnelling times and their relationship, including our definition of reflection time in (4.3) and relationship in (5.2), could be compared with experimental data to see which, if any, are confirmed.

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